

Vacuum fluctuations of a massless spin- $\frac{1}{2}$ field around multiple cosmic strings

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We study the interaction of a massless quantized spinor field with the gravitational field of N parallel static cosmic strings by using a perturbative approach. We show that the presence of more than one cosmic string gives rise to an additional contribution to the energy density of vacuum fluctuations, thereby leading to a vacuum force of attraction between two parallel cosmic strings.

1 Introduction

Cosmic strings predicted in the framework of various gauge theories with spontaneously broken symmetries could have been created at cosmological phase transitions in the early universe [1, 2]. Since cosmic strings are produced at very large energy scales, one might expect a highly curved spacetime around them. However in the case of a static and straight-line cosmic string there exists a very simple exact solution of the Einstein field equations, that describes a locally flat conical spacetime around the string [3]. Furthermore, it has been shown that one may also construct an appropriate exact solution of the Einstein equations for a snapping cosmic string, which serves as a source for spherical impulsive gravitational waves [4].

Although the locally flat structure of the spacetime implies that straight-line cosmic strings will not exert any local gravitational force on surrounding particles, the particles “interact” with the global conical structure. It gives rise to the distinctive gravitational effects, such as lensing of distant objects, conical bremsstrahlung, etc. [5, 6]. On the other hand, it is well known that the non-trivial topological structures restrict the modes of quantized fields propagating in locally flat spacetimes, thereby providing the appearance of vacuum boundary effects [7]. One may, therefore, consider the conical spacetime around static and straight-line cosmic strings as an attractive model for investigation the influence of gravitational field on the behaviour of quantized fields. In particular, such investigations for a massless quantized fields of different spins, propagating in the spacetime of a static cosmic string have been carried out in [8-10]. The propagation of a massless quantized scalar field in the spacetime of a snapping cosmic string was studied in [11].

Recently in [12] we calculated the effects of vacuum fluctuations of a massless quantized electromagnetic field propagating in the spacetime of multiple cosmic strings represented by N parallel static (fixed), straight-line strings. It has been shown that the presence of more than one cosmic string provides an additional contribution to the energy density of vacuum fluctuations, which results in the Casimir-like force of attraction between two parallel cosmic strings.

The aim of the present paper is to extend the calculations of [12] to the case of a massless quantized spinor field. First, for an instructive purpose, we shall rederive the expression for the vacuum expectation value of the energy-momentum tensor of a massless spin- $\frac{1}{2}$ field around a single cosmic

string. For the case of multiple cosmic strings we shall adopt a perturbative approach, in which the gravitational field of the strings is treated as small metric perturbations about the Minkowskian spacetime. We construct, at first-order in metric perturbations, the Feynman propagator for the spinor field and evaluate the energy density arising from vacuum fluctuations. We shall also show that at second-order perturbations of the metric, the effect of vacuum polarization of the spinor field gives rise to a force of attraction between two parallel static and straight-line cosmic strings. Throughout the paper we use geometrical units, in which $G = c = 1$ and $\hbar \approx 2.612 \times 10^{-66} cm^2$.

2 A single cosmic string

The metric of a static and straight-line cosmic string lying along the z-axis of the cylindrical coordinate system is given by the interval

$$ds^2 = dt^2 - dz^2 - dr^2 - b^2 r^2 d\theta^2 \quad (1)$$

where $b = 1 - 4\mu$, and μ is the linear mass density of the string. This metric is locally Minkowskian that is readily seen by passing to a new angular coordinate $\varphi \rightarrow b\theta$. We have

$$ds^2 = dt^2 - dz^2 - dr^2 - r^2 d\varphi^2 \quad (2)$$

However the global structure of this metric is a conical, as for the new azimuthal angle φ we have the following range

$$0 \leq \varphi < 2\pi(1 - 4\mu).$$

The presence of such a “deficit” in the azimuthal angle provides boundary conditions for quantized fields in the metric (2), which result in Casimir-like distortions of the spectrum of vacuum fluctuations. The effect of vacuum fluctuations for a massless spin- $\frac{1}{2}$ field has been calculated in [9] by using the method of Green’s functions. Here we shall briefly reproduce this result using a different approach, involving the summation of the exact field modes.

The propagation of a massless spin- $\frac{1}{2}$ field in curved space-time is governed by the Dirac equation

$$i \gamma^\mu \nabla_\mu \psi = 0 \quad (3)$$

where the γ^μ matrices satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}. \quad (4)$$

The spinor covariant derivative operator is defined as

$$\nabla_\mu = \partial_\mu - \Gamma_\mu$$

where Γ_μ is the spinor connection. Let us consider a tetrad of vector field e_a^μ , satisfying the relations

$$\eta_{ab} = g_{\mu\nu} e_a^\mu e_b^\nu, \quad e_a^\mu e_b^\nu \eta_{ab} = g^{\mu\nu} \quad (5)$$

For the sake of further convenience we shall use the Newman-Penrose tetrad of null vectors [13]

$$e_a^\mu = \{l^\mu, n^\mu, m^\mu, m^{*\mu}\} \quad (6)$$

which satisfy the normalization conditions $l_\mu n^\mu = 1$ and $m_\mu m^{*\mu} = -1$. The symbol $*$ denotes a complex conjugation. It is clear that with the choice of the tetrad (6) the symmetric constant matrices η_{ab} and η^{ab} have the only nonvanishing components $\eta_{12} = \eta_{21} = -\eta_{34} = -\eta_{43} = 1$, $\eta_{ab} = \eta^{ab}$. Since the ordinary constant Dirac matrices γ^a satisfy the condition

$$\{\gamma^a, \gamma^b\} = 2 \eta^{ab}, \quad (7)$$

then the coordinate-dependent matrices γ^μ obeying the relation (4) will be defined through the above introduced tetrad e_a^μ by

$$\gamma^\mu = e_a^\mu \gamma^a \quad (8)$$

while the spinor connection Γ_μ is defined as

$$\Gamma_\mu = \frac{1}{4} \gamma^a \gamma^b e_{a;\mu}^\nu e_{b\nu} \quad (9)$$

where the semicolon denotes the covariant differentiation with respect to the metric $g^{\mu\nu}$. The explicit form of the γ^μ matrices written in terms of the null vectors is

$$\gamma^\mu = \sqrt{2} \begin{pmatrix} 0 & 0 & n^\mu & -m^{*\mu} \\ 0 & 0 & -m^\mu & l^\mu \\ l^\mu & m^{*\mu} & 0 & 0 \\ m^\mu & n^\mu & 0 & 0 \end{pmatrix} \quad (10)$$

Turning to the cosmic string metric (2) one may choose the following null vectors

$$\begin{aligned} l^\mu &= \frac{1}{\sqrt{2}} (1, -1, 0, 0) & n^\mu &= \frac{1}{\sqrt{2}} (1, 1, 0, 0) \\ m^\mu &= \frac{1}{\sqrt{2}} (0, 0, -1, -\frac{i}{r}) & m^{*\mu} &= \frac{1}{\sqrt{2}} (0, 0, -1, \frac{i}{r}) \end{aligned} \quad (11)$$

Let us now consider the four component spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad (12)$$

then the Dirac equation (3) projected onto the null vectors (11) is decomposed into the two independent pairs of equations

$$\begin{aligned} D \psi_1 + (\delta^* + \beta) \psi_2 &= 0 \\ \Delta \psi_2 + (\delta + \beta) \psi_1 &= 0 \end{aligned} \quad (13)$$

and

$$\begin{aligned} \Delta \psi_3 - (\delta^* + \beta) \psi_4 &= 0 \\ D \psi_4 - (\delta + \beta) \psi_3 &= 0 \end{aligned} \quad (14)$$

where

$$\beta = \frac{1}{2} m^{*\mu}{}_{;\nu} m_\mu m^\nu = -\frac{1}{2\sqrt{2}r}$$

and

$$\begin{aligned} D &= l^\mu \partial_\mu = \frac{1}{\sqrt{2}} (\partial_t - \partial_z) & \Delta &= n^\mu \partial_\mu = \frac{1}{\sqrt{2}} (\partial_t + \partial_z) \\ \delta &= m^\mu \partial_\mu = -\frac{1}{\sqrt{2}} (\partial_r + \frac{i}{r} \partial_\varphi) \end{aligned} \quad (15)$$

are the directional derivative operators. Combining the equations (13) and (14) in an appropriate way we obtain the decoupled set of equations

$$\square_s \psi_s = 0 \quad (16)$$

where we have introduced the spin-weighted operator

$$\square_s = \partial_t^2 - \partial_z^2 - \frac{1}{r} \partial_r (r \partial_r) - \frac{1}{r^2} \partial_\varphi^2 - \frac{2is}{r^2} \partial_\varphi + \frac{s^2}{r^2}$$

with the spin-weight $s = \pm \frac{1}{2}$ and the spin-weighted modes

$$\psi_{(s=+\frac{1}{2})} = \begin{pmatrix} \psi_2 \\ \psi_4 \end{pmatrix}, \quad \psi_{(s=-\frac{1}{2})} = \begin{pmatrix} \psi_1 \\ \psi_3 \end{pmatrix}$$

The equations (16) may be solved by the separation of variables and the regular at $r = 0$ solutions have the form

$$\psi_s \sim e^{-i(\omega t - k z - m \nu \varphi)} J_{|m\nu+s|}(p r) \quad (17)$$

where $J(p r)$ is a Bessel function and $p^2 = \omega^2 - k^2$, $\nu = 1/b$. Since the transformation properties of spinors implies that a 2π rotation of a spinor changes its sign [14], the single-valuedness of the solutions (17) requires that $m = n - \frac{1}{2}$, or $m = n + \frac{1}{2}$ with $n = 0, \pm 1, \pm 2, \dots$. Having in hand these modes one can easily construct the complete sets of positive- and negative-frequency solutions u_{smkp} and v_{smkp} of the Dirac equation, which satisfy the normalization conditions

$$\begin{aligned} \int \bar{u}_{s'm'k'p'}(x) \gamma^0 u_{smkp}(x) \sqrt{-g} d^3 x &= \int \bar{v}_{s'm'k'p'}(x) \gamma^0 v_{smkp}(x) \sqrt{-g} d^3 x \\ &= \delta_{ss'} \delta_{mm'} \delta(k - k') \frac{\delta(p - p')}{\sqrt{p p'}} \end{aligned} \quad (18)$$

where $\bar{u} = u^\dagger \gamma^0$ and $\bar{v} = v^\dagger \gamma^0$ are the adjoint spinors.

The canonical quantization are performed in a usual way [7], by expanding the field operator $\hat{\psi}(x)$ on the complete sets of positive and negative frequency modes

$$\hat{\psi}(x) = \sum_{s,m} \int_{-\infty}^{\infty} dk \int_0^{\infty} dp p [\hat{a}_{smkp} u_{smkp}(x) + \hat{b}_{smkp}^\dagger v_{smkp}(x)] \quad (19)$$

where \hat{a}_{smkp} represents annihilation operator for particles and \hat{b}_{smkp}^\dagger is the creation operator for antiparticles. It should be noted that as the spacetime is flat everywhere outside cosmic strings one can define a vacuum state by choosing positive frequency modes with respect to the timelike Killing vector $\partial/\partial t$ of the flat spacetime.

The energy-momentum tensor for spin- $\frac{1}{2}$ has the form

$$T_{\mu\nu}(x) = \frac{i}{2} [\bar{\psi} \gamma_{(\mu} \nabla_{\nu)} \psi - (\nabla_{(\mu} \bar{\psi}) \gamma_{\nu)} \psi] \quad (20)$$

The vacuum expectation value of this quantity can be evaluated by representing it as a bilinear function of the fields and performing a renormalization procedure at the coincidence points $x = x'$. Substituting the expansion (19) into the equation (20) we obtain that its vacuum averaged $\langle T_{00}(x) \rangle$ component are reduced to the form

$$\langle T_{00}(x) \rangle = \lim_{x \rightarrow x'} (\partial_t^2 - \partial_t \partial_{t'}) \frac{U(x, x')}{2\omega} \quad (21)$$

where the two-points function $U(x, x')$ is

$$\begin{aligned} \frac{U(x, x')}{2\omega} &= \frac{\nu}{8\pi^2} \sum_m e^{i\nu m(\varphi - \varphi')} \int_0^\infty dp p \int_{-\infty}^\infty dk \frac{e^{-i\sqrt{k^2 + p^2} \tau + ik \zeta}}{\sqrt{k^2 + p^2}} \\ &\quad [J_{|m\nu + \frac{1}{2}|}(pr) J_{|m\nu + \frac{1}{2}|}(pr') + J_{|m\nu - \frac{1}{2}|}(pr) J_{|m\nu - \frac{1}{2}|}(pr')] \end{aligned} \quad (22)$$

and $\tau = t - t'$, $\zeta = z - z'$. The integrals over k and p in this expression are evaluated by means of the corresponding formulae given in [17]. As a result we have

$$\frac{U(x, x')}{2\omega} = \frac{\nu}{8\pi^2 r r'} \frac{1}{\sqrt{u^2 - 1}} \sum_{n=-\infty}^\infty e^{-i\nu(n - \frac{1}{2})(\varphi - \varphi')} \left(\xi^{-|\nu(n - \frac{1}{2}) + \frac{1}{2}|} + \xi^{-|\nu(n - \frac{1}{2}) - \frac{1}{2}|} \right) \quad (23)$$

where

$$u = \frac{r^2 + r'^2 + \zeta^2 - \tau^2}{2 r r'}, \quad \xi = u + \sqrt{u^2 - 1}$$

and we have taken $m = n - \frac{1}{2}$. The renormalization of the function $U(x, x')$ is achieved by subtracting from this expression its pure Minkowskian value, i.e at $\nu = 1$. Since the equation (21) involves only the derivatives over t and t' one can put $\varphi = \varphi'$, $z = z'$ and $r = r'$. Then the evaluation of the sum in (23) is significantly simplified and we have

$$\sum_{n=-\infty}^\infty \left(\xi^{-|\nu(n - \frac{1}{2}) + \frac{1}{2}|} + \xi^{-|\nu(n - \frac{1}{2}) - \frac{1}{2}|} \right) = 2 \frac{\xi^{-\frac{\nu}{2}} (\xi^{-\frac{1}{2}} + \xi^{\frac{1}{2}})}{1 - \xi^{-\nu}} \quad (24)$$

The further calculation of the renormalized quantity $U(x, x')$ becomes straightforward and substituting the result into the equation (21) we arrive at the following expression for the energy density of vacuum fluctuations

$$\langle T_{00}(x) \rangle = -\frac{\hbar}{2880} \frac{1}{\pi^2 r^4} (\nu^2 - 1) (7\nu^2 + 17) \quad (25)$$

This result is in agreement with that of given in [9]. The other components of $\langle T_{\mu\nu}(x) \rangle$ can be found by using the symmetry properties of the space-time (1) along with the conditions $T_\mu^\mu = 0$ and $T^{\mu\nu}_{;\nu} = 0$.

3 Multiple cosmic strings

If one makes a transformation of the radial coordinate $r \rightarrow r_i^b/b$, then the metric (1) can be transformed into the the following form

$$ds^2 = dt^2 - dz^2 - e^{-2\Lambda(x,y)}(dx^2 + dy^2) \quad (26)$$

where

$$\Lambda(x, y) = \sum_{i=1}^N 4\mu_i \ln r_i \quad r_i = [(x - \alpha_i)^2 + (y - \beta_i)^2]^{1/2} \quad (27)$$

It turns out that this metric is the exact solution of the Einstein equations [15], describing the space-time around N parallel static cosmic strings, passing through the points $x_i = (\alpha_i, \beta_i)$. The corresponding tetrad of null vectors for the metric (26) can be chosen as

$$\begin{aligned} l^\mu &= \frac{1}{\sqrt{2}} (1, -1, 0, 0) & n^\mu &= \frac{1}{\sqrt{2}} (1, 1, 0, 0) \\ m^\mu &= \frac{1}{\sqrt{2}} e^\Lambda (0, 0, -1, -i) & m^{*\mu} &= -\frac{1}{\sqrt{2}} e^\Lambda (0, 0, 1 - i) \end{aligned} \quad (28)$$

Using these vectors in the equations (8) and (9) we find that

$$\gamma^0(x) = \gamma^0, \quad \gamma^3(x) = \gamma^3, \quad \gamma^1(x) = e^\Lambda \gamma^1, \quad \gamma^2(x) = e^\Lambda \gamma^2, \quad (29)$$

where $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ are ordinary constant Dirac matrices, and the only non-vanishing components of the spinor connection Γ_μ are

$$\Gamma_1 = -\frac{1}{2} \gamma^1 \gamma^2 e^\Lambda \partial_y \Lambda \quad \Gamma_2 = -\frac{1}{2} \gamma^2 \gamma^1 e^\Lambda \partial_x \Lambda \quad (30)$$

Now we shall evaluate the vacuum expectation value of the energy-momentum tensor (20) in the metric (26). We start with the expression

$$\langle T_{\mu\nu}(x) \rangle = -\frac{\hbar}{4} \lim_{x \rightarrow x'} \text{Tr} [\gamma_\mu (\nabla_\nu - \nabla_{\nu'}) + \gamma_\nu (\nabla_\mu - \nabla_{\mu'})] S_F(x, x') \quad (31)$$

where

$$S_F(x, x') = -i \langle 0 | T (\bar{\psi}(x') \psi(x)) | 0 \rangle \quad (32)$$

is the Feynman propagator which obeys the equation

$$i\gamma^\mu (\partial_\mu - \Gamma_\mu) S_F(x, x') = \frac{1}{\sqrt{-g}} \delta^4(x - x') \quad (33)$$

The substitution of the relations (29) and (30) into this equation enables us to cast it in the form

$$i\gamma^a \partial_a S_F(x, x') = \delta^4(x - x') + V S_F(x, x') \quad (34)$$

where γ^a once again denotes the flat space-time Dirac matrices,

$$V = i [(1 - e^{-2\Lambda}) \gamma^A \partial_A + (1 - e^{-\Lambda}) \gamma^\alpha \partial_\alpha + \frac{1}{2} e^{-\Lambda} \gamma^\alpha \partial_\alpha \Lambda] \quad (35)$$

and the index A takes the values $(0, 3) \equiv (t, z)$, while $\alpha = (1, 2) \equiv (x, y)$. In order to construct the solution of equation (34) we use a perturbative approach. For this purpose, we assume that the linear mass densities of the cosmic strings are sufficiently small, ($\mu_i \ll 1$), which is indeed the case for realistic cosmic strings ($\mu_i \approx 10^{-6}$). Then the metric (26) can be expanded in powers of μ_i , about a fixed flat background, and the potential (35) can be considered as a small perturbing term in the equation (34). This approximation allows us to write the solution of equation (34) as perturbation series

$$S_F(x, x') = S_F^{(0)} + S_F^{(0)} V S_F^{(0)} + S_F^{(0)} V S_F^{(0)} V S_F^{(0)} + \dots \quad (36)$$

The zeroth order free Feynman propagator is defined as

$$S_F^{(0)}(x, x') = \int \frac{d^4 k}{(2\pi)^4} \frac{\gamma k}{k^2} e^{-ik(x-x')} \quad (37)$$

where $\gamma k = \gamma^a k_a$. The successive terms in the expansion (36) correspond to the higher order in μ_i contributions to the free propagator. Expanding the

equation (35) in powers of μ_i and taking it into account in (36) we obtain that the first order corrections to the free Feynman propagator are given by

$$S_F^{(1)}(x, x') = \frac{1}{64\pi^6} \int d^4k \delta(k_0) \delta(k_3) \Lambda(k) e^{-ikx'} \int \frac{d^4q}{q^2 (q-k)^2} \gamma q$$

$$[\gamma^A q_A + \gamma q - \frac{1}{2} \gamma k] \gamma(q-k) e^{-iq(x-x')} \quad (38)$$

In order to evaluate the finite vacuum expectation value of the energy-momentum tensor(31) one needs to regularize it, for which we shall use the dimensional regularization procedure [16]. The latter gives to the the vanishing value for the zeroth order propagator [7], however, substituting the first-order propagator (38) into the expression (31), for its $\langle T_{00}^{(1)}(x) \rangle$ component at the coincidence limit $x \rightarrow x'$ we find

$$\langle T_{00}^{(1)}(x) \rangle = \frac{i\hbar}{64\pi^6} \text{Tr} \int d^4k \delta(k_0) \delta(k_3) \Lambda(k) e^{-ikx'} \int \frac{d^4q}{q^2 (q-k)^2}$$

$$\gamma^0 q_0 \gamma q [\gamma^A q_A + \gamma q - \frac{1}{2} \gamma k] \gamma(q-k) \quad (39)$$

In the framework of the dimensional regularization procedure the integral over q is calculated, by performing an analytical continuation to d dimensional space and then the result is expanded about $d = 4 - \epsilon$, ($\epsilon \rightarrow 0$) [14]. Having done all these we obtain

$$\langle T_{00}^{(1)}(x) \rangle = - \frac{\hbar}{240\pi^3} \Gamma(-1 + \frac{\epsilon}{2}) \sum_{i=1}^N \mu_i \int d^2k (k^2)^{1-\frac{\epsilon}{2}} e^{-ikr_i} \quad (40)$$

In obtaining of this expression we have used the explicit form for the Fourier component of the function $\Lambda(x, y)$

$$\Lambda(k) = 8\pi \sum_{i=1}^N \mu_i \frac{e^{ikx_i}}{k^2} \quad (41)$$

It is important to stress that the expression (40) is finite, as the divergent terms are exactly cancelled by one another. Indeed, using the integral [12]

$$\int_0^\infty d^d k (k^2)^\nu e^{-ikR} = \pi^{d/2} \frac{2^{2\nu+d}}{R^{2\nu+d}} \frac{\Gamma(\nu + d/2)}{\Gamma(-\nu)} \quad (42)$$

in Eq.(40), in which $d = 2 - \epsilon$, and $\nu = 1 - \frac{\epsilon}{2}$, we finally obtain the following result

$$\langle T_{00}^{(1)}(x) \rangle = -\frac{\hbar}{15\pi^2} \sum_{i=1}^N \frac{\mu_i}{r_i^4} \quad (43)$$

It is seen from this expression that at first order in μ_i , the contributions to the energy density of vacuum fluctuations of spin- $\frac{1}{2}$ field around N parallel static cosmic strings are linearly summed. One can easily see that in the case of a single cosmic string the expression (43) coincides with (25) taken at $\mu \ll 1$.

4 Vacuum force between two parallel cosmic strings

Let us now proceed to the second-order metric contributions to the vacuum expectation value of the energy-momentum tensor (31). In analogy with the case of electromagnetic fluctuations [12] we shall show that the energy density of vacuum fluctuations of a massless spinor field involves a term which depends upon the separation distance between cosmic strings, therefore, produces an attractive force between the strings.

Using the expressions (35) and (37) in the expansion (36) we find that the second-order metric corrections to the free Feynman propagator have the form

$$\begin{aligned} S_F^{(2)}(x, x) = & \frac{1}{256\pi^8} \int d^4k \delta(k_0) \delta(k_3) \Lambda(k) e^{-ikx'} \int d^4p \delta(p_0) \delta(p_3) \Lambda(p) e^{-ipx'} \\ & \int \frac{d^4q}{q^2} \frac{\gamma q (\gamma^A q_A + \gamma q - \frac{1}{2} \gamma k) \gamma(q - k)}{(q - k)^2 (q - k - p)^2} \\ & [\gamma^B q_B + \gamma(q - k - p) + \frac{1}{2} \gamma p] \gamma(q - k - p) e^{-iq(x-x')} \end{aligned} \quad (44)$$

Substituting this expression into the $\langle T_{00} \rangle$ component of Eq. (31) at the coincidence limit $x = x'$, we calculate the traces of γ matrices using the well-known theorems [14], then taking into account the relation (41) we arrive at the expression

$$\langle T_{00}^{(2)}(x) \rangle = \frac{i\hbar}{4\pi^6} \sum_{i=1}^N \mu_i \sum_{j=1}^N \mu_j \int \frac{d^4k}{k^2} \delta(k_0) \delta(k_3) e^{-ik(x-x_i)}$$

$$\int \frac{d^4 p}{p^2} \delta(p_0) \delta(p_3) e^{-ip(x-x_j)} \int \frac{d^4 q}{q^2} \frac{q_0^2}{(q-k)^2(q-k-p)^2} \mathcal{N}(q, k, p) \quad (45)$$

where

$$\begin{aligned} \mathcal{N}(q, k, p) = & 4(q_A q^A)[4 q_A q^A + (q-k)^2 + (k+p)^2 - kp] + 2(q^2 - 2 kq)(kq - 2q^2) \\ & - k^2(2q^2 + 2kq - p^2 - 2k^2) - (kp)(q^2 - 3k^2 + 2kq) + 2(pq)(q-k)(3q-k) \end{aligned}$$

We note that in this expression we are interested only in contributions to the vacuum energy density which are proportional to the products of the linear mass densities of different cosmic strings. It is clear that the latter describes the energy density of vacuum interaction between the strings. As for the remaining contributions, they form higher order corrections to the vacuum energy density given by (43). However first we need to regularize the expression (45), for which we again use the dimensional regularization procedure. We evaluate the integral over q by performing an analytical continuation to $d = 4 - \epsilon$ dimensions using the scheme, described in [14]. After some straightforward algebra, we arrive at the following result

$$\begin{aligned} \langle T_{00}^{(2)}(x) \rangle = & -\frac{\hbar}{4\pi^4} \sum_{i=1}^N \mu_i \sum_{j=1}^N \mu_j \int \frac{d^2 k}{k^2} e^{-ik(x-x_i)} \int \frac{d^2 p}{p^2} e^{-ip(x-x_j)} \\ & \int_0^1 dz_1 \int_0^{1-z_1} dz_2 [40 \Gamma(-2 + \frac{\epsilon}{2}) B^{2-\frac{\epsilon}{2}} + C_1 \Gamma(-1 + \frac{\epsilon}{2}) B^{1-\frac{\epsilon}{2}} \\ & + C_2 \Gamma(\frac{\epsilon}{2}) B^{-\frac{\epsilon}{2}}] \end{aligned} \quad (46)$$

where we have introduced

$$\begin{aligned} C_1 = & -12Q(Q - k - p) + 4(k^2 + p^2 + \frac{1}{8}kp) \\ C_2 = & Q^2(-2Q^2 + 5kQ - k^2 - \frac{1}{2}kp + 3pQ) \\ & - kQ(k^2 + kp + 2kQ + 4pQ) + \frac{1}{2}k^2(p^2 + 2k^2 + 3kp + 2pQ) \end{aligned}$$

and

$$Q_\alpha = k_\alpha z_1 + (k_\alpha + p_\alpha) z_2$$

$$B = k^2 z_1(1 - z_1) + (k + p)^2 z_2(1 - z_2) - 2k(k + p) z_1 z_2$$

The energy of vacuum fluctuations per unit length of the strings may be evaluated by means of the formula

$$E = \int d^2x \sqrt{-g} \langle T_{00}(x) \rangle \quad (47)$$

which at the second order metric perturbations takes the following form

$$E = \int dx dy \left[\langle T_{00}^{(2)}(x) \rangle - 2\Lambda(x) \langle T_{00}^{(1)}(x) \rangle \right] \quad (48)$$

For the sake of certainty let us consider two parallel cosmic strings. Using the relations (40) and (46) in the equation (48) we first carry out the integration over x and y , then the calculations of remaining integrals become straightforward and keeping only the terms involving the product of the linear mass densities of the cosmic strings we find

$$E_{int} = -\frac{\hbar}{15\pi^2} \mu_1 \mu_2 \Gamma\left(\frac{\epsilon}{2}\right) \int d^2k (k^2)^{-\frac{\epsilon}{2}} e^{-ika}. \quad (49)$$

It should be stressed that this quantity is finite as the involved divergent at $\epsilon \rightarrow 0$ terms are compensated by one another. Indeed taking the double integration over k using the integral (42) we obtain the expression

$$E_{int} = -\frac{4\hbar}{15\pi} \frac{\mu_1 \mu_2}{a^2} \quad (50)$$

where a is the separation distance between the cosmic strings. It is clear that the presence of this energy, will produce an attractive force per unit length of the strings given by

$$F = -\frac{8\hbar}{15\pi} \frac{\mu_1 \mu_2}{a^3} \quad (51)$$

As we have already mentioned above the static and straight-line cosmic strings do not exert any local gravitational force on surrounding matter. Here we have shown that the propagation of a massless quantized spinor field in the spacetime of more than one cosmic string induces a force of attraction (51)

between two cosmic strings, which falls off as the third power of the separation distance. The reason for this is the restriction of the modes of quantized field by the multiconical structure of the spacetime around the cosmic strings. Unlike the case of a single cosmic string, it is difficult to construct the exact modes of the field equations in the metric of multiple cosmic strings, so we have used a perturbative approach along with the dimensional regularization procedure. We note that the expression (51) coincides with the corresponding result for a massless scalar field [18], while 2 times smaller than the result for a massless vector field [12].

It is important to stress that the above result is obtained within the one-loop approximation and therefore holds provided that the separation distance between the cosmic strings is much greater than the typical thicknesses of their cores.

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